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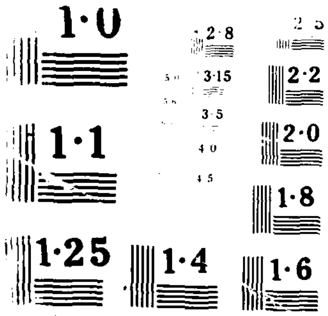
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20. ABSTRACT CONTINUED

Let $X = \{X(t) : t \geq 0\}$ be an irreducible semi-Markov process (SMP) on countable state space E. For fixed $z \in E$, let $T(z) = \inf\{t \geq 0 : X(t-) \neq z, X(t) = z\}$ and set $Y(f) = \int_0^{T(z)} f(X(t))\,dt$, where $f : E \to R$ is an arbitrary function. Our objective is to study the mixed moments of the form $E\prod_{i=1}^{r} Y(f_i)$, when $f_i : E \to R$ is an arbitrary function, for $i = 1, 2, \ldots, r$, and r is a positive integer. This quantity is especially relevant to the regenerative simulation. Also, several useful variations and generalizations are introduced and studied.

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by

Chia-Hon Chien

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ABSTRACT

Let $X = \{X(t) : t \geq 0\}$ be an irreducible semi-Markov process (SMP) on countable state space E. For fixed $z \in E$, let $T(z) = \inf\{t \geq 0 : X(t-) \neq z, X(t) = z\}$ and set $Y(f) = \int_0^{T(z)} f(X(t))\,dt$, where $f : E \to R$ is an arbitrary function. Our objective is to study the mixed moments of the form $E\prod_{i=1}^{r} Y(f_i)$, when $f_i : E \to R$ is an arbitrary function, for $i = 1, 2, \ldots, r$, and r is a positive integer. This quantity is especially relevant to the regenerative simulation. Also, several useful variations and generalizations are introduced and studied.

Keywords: semi-Markov processes

regenerative simulation

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Recursive Moment Formulas for Semi-Markov Processes

by

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1. Introduction

Let $X = \{X(t) : t \geq 0\}$ be an irreducible semi-Markov process (SMP) on countable state space E. For fixed $z \in E$, let $T(z) = \inf\{t \geq 0 : X(t-) \neq z, X(t) = z\}$ and set

$$Y(f) = \int_0^{T(z)} f(X(t))\,dt,$$

where $f : E \to R$ is an arbitrary function. (Hereafter, we will suppress the z in $T(z)$ when no confusion is likely.) Our objective is to study the mixed moments of the form

$$E\prod_{i=1}^{r} Y(f_i),$$

when $f_i : E \to R$ is an arbitrary function, for $i = 1, 2, \ldots, r$, and r is a positive integer.

Hordijk, Iglehart, and Schassberger (1976) showed how to do this for $r \leq 2$, when X is a discrete time Markov chain (DTMC) or a continuous time Markov chain (CTMC) with countably many states. Glynn and Iglehart (1984) showed how to calculate such quantities when X is an SMP with countably many states, but with the restriction that $f_i = g, i = 1, \ldots, m$ and $f_i = h, i = m+1, \ldots\ r$ for some g, h and $m \leq r$. In this report, we will show how to obtain the same quantities for the more general cases, namely, without the restriction that only two different f_i's can appear in $E\prod_{i=1}^{r} Y(f_i)$. By exploiting some combinatorial relations, we will prove three recursive moment formulas, one for SMP, one for DTMC, and one for CTMC.

We now outline the material to be covered in the following sections. We begin in Section 2 by defining the notations. In Section 3 we will explore some combinatorial relations and obtain a recursive formula of $E\prod_{i=1}^{r} Y(f_i)$ for the SMP case. In Section 4 and Section 5, we will prove two more recursive moment formulas, one for DTMC, and one for CTMC. Finally, in Section 6, we will briefly discuss some generalizations.

2. Notations

To state our result, denote $Q = \{Q(x, y, t) : x, y \in E\}$ to be the usual semi-Markov kernel, $P = (P_{xy} : x, y \in E)$ to be the transition matrix of the underlying Markov chain $R = \{R_n : n \geq 0\}$ of X; and let G_n and β_n be a matrix and a function, respectively, defined by (We follow the notation given in Glynn and Iglehart (1984).)

$$G_n(x, y) = \begin{cases} P_{xy}\mu_n(x, y), & \text{if } y \neq x; \\ 0, & \text{if } y = x; \end{cases}$$

and

$$\beta_n(x) = \sum_{y \in E} \mu_n(x, y) P_{xy},$$

where

$$\mu_n(x, y) = \int_0^\infty t^n F(x, y, dt),$$

and $F(x, y, t) = Q(x, y, t)/P_{xy}$. For convenience, we assume that $F(x, y, 0) = 0$, for all $x, y \in E$.

Following Hordijk, Iglehart, and Schassberger (1976), we consider vectors such as $(v(0), v(1), \ldots, v(k))$ to be column vectors. In addition, for vectors u and v, the symbol $u \circ v$ denotes the Hadamard product of vectors

$$(u(0)v(0), u(1)v(1), \ldots, u(k)v(k));$$

for a matrix $A = (a_0, a_1, \ldots, a_m)$, set

$$u \circ A = A \circ u = (u \circ a_0, u \circ a_1, \ldots, u \circ a_m);$$

and for a matrix $B = (b_0, b_1, \ldots, b_m)$, set

$$A \circ B = B \circ A = (a_0 \circ b_0, a_1 \circ b_1, \ldots, a_m \circ b_m).$$

Finally, for vectors $f_1, f_2, \ldots, f_r$, define

$$\bigcirc_{i=1}^{r} f_i = f_1 \circ f_2 \circ \cdots \circ f_r,$$

and set $u^0(\cdot) = (1, 1, \ldots, 1)$ and $u^{n+1} = u \circ u^n = u^n \circ u$ for $n \geq 0$.

We begin by noting

$$Y(f) = \int_0^{T(z)} f(X(t))\,dt$$

$$= \int_0^{\infty} f(X(t))\,1_{\{T>t\}}\,dt$$

$$= \sum_{n=0}^{\infty} \int_{\xi_n}^{\xi_{n+1}} f(X(t))\,1_{\{T>t\}}\,dt$$

$$= \sum_{n=0}^{\infty} f(R_n)\,1_{\{\delta>n\}}\,(\xi_{n+1} - \xi_n),$$

where ξ_n is the nth jump time of the SMP, $\xi_0 \equiv 0$; δ is the length of the first z-cycle for the underlying discrete time Markov chain R; $1_A(\omega) = 1, \omega \in A; 1_A(\omega) = 0, \omega \notin A$. Since $Y(f)$ can be written in the form as $\sum_{n=0}^{\infty}(\cdot)$, it follows that we can write $\prod_{i=1}^{r} Y(f_i) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\prod_{i=1}^{r} f_i(R_{n_i})(\xi_{n_i+1} - \xi_{n_i})1_{\{\delta>n_i\}}$.

To make inference of the above product form, we introduce the following notations. First of all, let $N \equiv \{1,2,3,\ldots\}$, $N_* \equiv \{0\} \cup N$, and $N_*^r \equiv \{(n_1,\ldots,n_r) : n_i \in N_*, 1 \le i \le r\}$ for each positive integer r. Then for $S_1, S_2, \ldots, S_t$ to be a partition of $N_r \equiv \{n : n \le r, n \in N\}$, define

$$[\langle n_1,\ldots,n_r\rangle] \equiv N_*^r,$$

$$[(n_1,\ldots,n_r)] \equiv \{(n_1,\ldots,n_r) \in N_*^r : n_1 = \cdots = n_r\},$$

$$[(n_i : i \in S_1) < \cdots < (n_i : i \in S_{t-1}) < \langle n_i : i \in S_t\rangle]$$
$$\equiv \{(n_1,\ldots,n_r) \in N_*^r : n_i = n_{S_k}, i \in S_k, 1 \le k \le t-1;$$
$$n_{S_1} < \cdots < n_{S_{t-1}} < n_i, i \in S_t;\ \textit{for some integers}\ n_{S_i}, 1 \le i \le t-1\};$$

$$[(n_i : i \in S_1) \le \cdots \le (n_i : i \in S_{t-1}) \le \langle n_i : i \in S_t\rangle]$$
$$\equiv \{(n_1,\ldots,n_r) \in N_*^r : n_i = n_{S_k}, i \in S_k, 1 \le k \le t-1;$$
$$n_{S_1} \le \cdots \le n_{S_{t-1}} \le n_i, i \in S_t;\ \textit{for some integers}\ n_{S_i}, 1 \le i \le t-1\};$$

and for arbitrary function $f : N_*^r \to R$, and arbitrary subset $A \subseteq N_*^r$, define

$$\bar{f}_A \equiv \sum_{(n_1,\ldots,n_r)\in A} f(n_1,\ldots,n_r),$$

for example,

$$\bar{f}_{[(n_1,\dots,n_r)]} \equiv \sum_{(n_1,\dots,n_r)\in[(n_1,\dots,n_r)]} f(n_1,\dots,n_r),$$

$$\tilde{f}_{[(n_1,\dots,n_r)]} \equiv \sum_{(n_1,\dots,n_r)\in[(n_1,\dots,n_r)]} f(n_1,\dots,n_r),$$

etc; and denote $|\bar{f}|_A \equiv \sum_{(n_1,\dots,n_r)\in A} |f(n_1,\dots,n_r)|$.

3. A recursive formula for semi-Markov processes

We proceed via a series of lemmas.

Lemma 1.

$$1_{[(n_1,\dots,n_r)]} = \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} 1_{[(n_i:i\in S)<(n_i:i\in N_r\setminus S)]} + 1_{[(n_1,\dots,n_r)]}.$$

Proof: To exploit the idea of the lemma, we only prove a special case that $r = 3$, while the general case follows easily by using the same argument.

For $r = 3$, the lemma reduces to

$$\begin{aligned} 1_{[(n_1,n_2,n_3)]} = {} & 1_{[n_1<(n_2,n_3)]} + 1_{[n_2<(n_1,n_3)]} + 1_{[n_3<(n_1,n_2)]} \\ & + 1_{[(n_1,n_2)<n_3]} + 1_{[(n_1,n_3)<n_2]} + 1_{[(n_2,n_3)<n_1]} \\ & + 1_{[(n_1,n_2,n_3)]}. \end{aligned}$$

To prove this, we have to show that the seven sets on RHS are non-overlapping, and the union of them equals the set on LHS. The first part is obvious, since the first (second, third, respectively) term on RHS means: among the parameters $\{n_1, n_2, n_3\}$, n_1 (n_2, n_3, respectively) is the unique, least element. While the fourth (fifth, sixth, respectively) term on RHS means: among the parameters $\{n_1, n_2, n_3\}$, $n_1 = n_2$ ($n_1 = n_3$, $n_2 = n_3$, respectively), and n_3 (n_2, n_1, respectively) is the unique, greatest number. And the last term on RHS means: $n_1 = n_2 = n_3$. Because these event are mutually exclusively, this proves the first part.

To prove the second part, we have prove that for any (n_1, n_2, n_3) such that $0 \le n_1, n_2, n_3 < \infty$, it must belong to at least one of the seven sets in RHS. This should be obvious, since among any three numbers, either there exists a unique, least number; or two of the numbers are equal, and they are strictly smaller than the third one; or otherwise, the three numbers must be equal. This completes the proof.

Corollary 1.

$$
\begin{aligned}
&1_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]} \\
&= \sum_{j=1}^{|S_t|-1} \sum_{\substack{S:|S|=j \\ S\subsetneq S_t}} 1_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S)<(n_i:i\in S_t\setminus S)]} \\
&\qquad + 1_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}.
\end{aligned}
$$

Proof: Exactly as Lemma 1.

Corollary 2. *If* $|\tilde{f}|_{[(n_1,\ldots,n_r)]} = \sum_{(n_1,\ldots,n_r)\in N_*^r} |f(n_1,\ldots,n_r)| < \infty$ *then*

$$
\tilde{f}_{[(n_1,\ldots,n_r)]} = \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} \tilde{f}_{[(n_i:i\in S)<(n_i:i\in N_r\setminus S)]} + \tilde{f}_{[(n_1,\ldots,n_r)]}.
$$

Proof: From Lemma 1 and the absolute summability assumption.

Corollary 3. *If* $|\tilde{f}|_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]} < \infty$, *then*

$$
\begin{aligned}
&\tilde{f}_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]} \\
&= \sum_{i=1}^{|S_t|-1} \sum_{\substack{S:|S|=j \\ S\subsetneq S_t}} \tilde{f}_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S)<(n_i:i\in S_t\setminus S)]} \\
&\qquad + \tilde{f}_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}.
\end{aligned}
$$

Proof: Using Corollary 1.

For fixed f_i, $i = 1,\ldots,r$, and fixed $A \subseteq N_*^r$, we define L as

$$
L_A \equiv E\left\{ \sum_{(n_1,\ldots,n_r)\in A} \left[\prod_{i=1}^{r} f_i(R_{n_i})(\xi_{n_i+1} - \xi_{n_i}) 1_{\{\delta > n_i\}}\right] \right\},
$$

where δ is the length of the first z-cycle for the embedded DTMC R.

Lemma 2. *If* $\sum_{(n_1,\ldots,n_r)\in N_*^r} E\{\prod_{i=1}^r [|f_i(R_{n_i})|(\xi_{n_i+1} - \xi_{n_i}) 1_{\{\delta > n_i\}}]\} < \infty$ *then*

$$
\begin{aligned}
E\prod_{i=1}^{r} Y(f_i) &= L_{[(n_1,\ldots,n_r)]} \\
&= \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} L_{[(n_i:i\in S)<(n_i:i\in N_r\setminus S)]} + L_{[(n_1,\ldots,n_r)]}.
\end{aligned}
$$

Proof: The first equality follows from the definition of L; the second equality follows from the absolute summability assumption and Corollary 2 of Lemma 1.

Corollary 1. *If*

$$\sum_{(n_1,\ldots,n_r)\in[(n_i:i\in S_1)< \quad <(n_i:i\in S_{t-1})<(n_i:i\in S_t)]} E\{\prod_{i=1}^{r}[|f_i(R_{n_i})|(\xi_{n_i+1}-\xi_{n_i})1_{\{\delta>n_i\}}]\}<\infty$$

then

$$\begin{aligned} &L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]} \\ &=\sum_{j=1}^{|S_t|-1}\sum_{\substack{S:|S|=j\\ S\subsetneq S_t}} L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S)<(n_i:i\in S_t\setminus S)]} \\ &\quad + L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}. \end{aligned}$$

Proof: By using Corollary 3 of Lemma 1, it can be proved in the same way as Lemma 2.

Lemma 3. *If* $\sum_{n=0}^{\infty} E\{[\prod_{i=1}^{r}|f_i(R_n)|](\xi_{n+1}-\xi_n)^r 1_{\{\delta>n\}}\}<\infty$, *and define* $EZ(f)\equiv\sum_{n=0}^{\infty}G_0^n f$, *then*

$$L_{[(n_1,\ldots,n_r)]}=\sum_{n=0}^{\infty}G_0^n(\otimes_{i=1}^{r}f_i\circ\beta_r)=EZ(\otimes_{i=1}^{r}f_i\circ\beta_r).$$

Moreover, the vector $L_{[(n_1,\ldots,n_r)]}$ *is the unique solution,* y, *of*

$$y=\otimes_{i=1}^{r}f_i\circ\beta_r+G_0 y$$

satisfying $G_0^n y\to 0$ *as* $n\to\infty$.

Proof: Observe that our absolute summability assumption will justify all the interchanges of E's, $\int$'s, and $\sum$'s. Since

$$\begin{aligned} &L_{[(n_1,\ldots,n_r)]} \\ &=E\left\{\sum_{n=0}^{\infty}[\prod_{i=1}^{r}f_i(R_n)](\xi_{n+1}-\xi_n)^r 1_{\{\delta>n\}}\right\} \\ &=\sum_{n=0}^{\infty}E\left\{[\prod_{i=1}^{r}f_i(R_n)]E[(\xi_{n+1}-\xi_n)^r\mid R_n]1_{\{\delta>n\}}\right\} \\ &=\sum_{n=0}^{\infty}E\left\{[\prod_{i=1}^{r}f_i(R_n)]\beta_r(R_n)1_{\{\delta>n\}}\right\}, \end{aligned}$$

and $E_l[1_{(R_n=j)}1_{(\delta>n)}] = G_0^n(l,j)$, thus

$$\sum_{n=0}^{\infty} E_l\left\{[\prod_{i=1}^{r} f_i(R_n)]\beta_r(R_n)1_{\{\delta>n\}}\right\}$$

$$=\sum_{n=0}^{\infty} E_l\left\{\sum_{j\in E}[\prod_{i=1}^{r} f_i(j)]\beta_r(j)1_{\{\delta>n\}}1_{(R_n=j)}\right\}$$

$$=\sum_{n=0}^{\infty}\sum_{j\in E} G_0^n(l,j)(\prod_{i=1}^{r} f_i(j)\circ\beta_r(j))$$

$$=[\sum_{n=0}^{\infty} G_0^n(\otimes_{i=1}^{r} f_i\circ\beta_r)]_l$$

$$=[EZ(\otimes_{i=1}^{r} f_i\circ\beta_r)]_l\,.$$

The first result follows immediately.

Ths proof of the second part of this theorem is very similar to Theorem (3.1) in Hordijk, Iglehart, and Schassberger (1976), and thus omitted here.

Remark: An immediate consequence of this lemma: If $\sum_{n=0}^{\infty} E\{|f(R_n)|(\xi_{n+1}-\xi_n)1_{\{\delta>n\}}\}<\infty$, then $EY(f) = L_{[n]} = \sum_{n=0}^{\infty} G_0^n[f\circ\beta_1] = EZ(f\circ\beta_1)$. This is the first part of Theorem (5.14) in Iglehart [3].

Remark: If E is finite, then $G_0^n \to 0$ as $n\to\infty$, thus the uniqueness is automatically satisfied, and also $EZ=\sum_{n=0}^{\infty} G_0^n = (1-G_0)^{-1}$, thus $L_{[(n_1,\ldots,n_r)]} = (1-G_0)^{-1}[\otimes_{i=1}^{r} f_i\circ\beta_r]$.

Lemma 4. *If* $\sum_{(n_1,\ldots,n_r)\in[(n_i:i\in S_1)<\cdots<(n_i:i\in S_t)]} E\{\prod_{i=1}^{r}[|f_i(R_{n_i})|(\xi_{n_i+1}-\xi_{n_i})1_{\{\delta>n_i\}}]\}<\infty$ *then,*

$$L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_t)]}$$

$$=\sum_{n_{S_1}=0}^{\infty} G_0^{n_{S_1}}(\otimes_{i\in S_1} f_i\circ G_{|S_1|}\sum_{n_{S_2}=0}^{\infty} G_0^{n_{S_2}}(\otimes_{i\in S_2} f_i\circ G_{|S_2|}\cdots\sum_{n_{S_t}=0}^{\infty} G_0^{n_{S_t}}(\otimes_{i\in S_t} f_i\circ\beta_{|S_t|})\cdots))$$

$$=EZ(\otimes_{i\in S_1} f_i\circ G_{|S_1|}EZ(\otimes_{i\in S_2} f_i\circ G_{|S_2|}\cdots EZ(\otimes_{i\in S_t} f_i\circ\beta_{|S_t|})\cdots)).$$

Proof: This lemma can be shown by using a similar path decomposition argument as in Lemma 3. We omit the details.

Remark: Lemma 3 is a special case of Lemma 4.

Remark: If E is finite, then $L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_t)]} = (1-G_0)^{-1}(\otimes_{i\in S_1} f_i \circ G_{|S_1|}(1-G_0)^{-1}(\otimes_{i\in S_2} f_i \circ G_{|S_2|}\cdots(1-G_0)^{-1}(\otimes_{i\in S_t} f_i \circ \beta_{|S_t|})\cdots))$.

Theorem 1. *For an irreducible SMP, and arbitrary integer r, if $\sum_{(n_1,\ldots,n_r)\in N_r} E\{\prod_{i=1}^{r}\|f_i(R_{n_i})\|(\xi_{n_i+1}-\xi_{n_i})1_{\{\delta>n_i\}}\}<\infty$ then,*

(1) for each $r'\le r$, and $EZ(f)\equiv\sum_{n=0}^{\infty}G_0^n f$, we have

$$E\{\prod_{i=1}^{r'}Y(f_i)\} = L_{[(n_1,\ldots,n_{r'})]}$$

$$=\sum_{j=1}^{r'-1}\sum_{\substack{S:|S|=j\\ S\subseteq N_{r'}}} EZ(\otimes_{i\in S}f_i\circ G_j E\{\prod_{i\in N_{r'}\setminus S}Y(f_i)\}) + EZ(\otimes_{i=1}^{r'}f_i\circ\beta_{r'})$$

$$=EZ\left(\sum_{j=1}^{r'-1}\sum_{\substack{S:|S|=j\\ S\subseteq N_{r'}}}\otimes_{i\in S}f_i\circ G_j E\{\prod_{i\in N_{r'}\setminus S}Y(f_i)\} + \otimes_{i=1}^{r'}f_i\circ\beta_{r'}\right);$$

(2) for arbitrary partition $S_1,\ldots,S_t (t\ge 2)$ of N_r,

$$L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}$$
$$=EZ(\otimes_{i\in S_1}f_i\circ G_{|S_1|}EZ(\otimes_{i\in S_2}f_i\circ G_{|S_2|}\cdots EZ(\otimes_{i\in S_{t-1}}f_i\circ G_{|S_{t-1}|}E\{\prod_{i\in S_t}Y(f_i)\})\cdots));$$

(3) for arbitrary nonempty proper subset S of N_r,

$$L_{[(n_i:i\in S)<(n_i:i\in N_r\setminus S)]} = EZ(\otimes_{i\in S}f_i\circ G_{|S|}E\{\prod_{i\in N_r\setminus S}Y(f_i)\}).$$

Remark: If $r'=1$, (1) becomes: $EY(f)=EZ(f\circ\beta_1)=\sum_{n=0}^{\infty}G_0^n(f\circ\beta_1)$; when $r'=2$, (1) becomes: $EY(f_1)Y(f_2) = EZ(f_1\circ G_1 EY(f_2)+f_2\circ G_1 EY(f_1)) + EY(f_1\circ f_2\circ\beta_2)$. These quantities agree with Theorem (5.14) in Iglehart [3].

Remark: If E is finite, in addition to the existing assertions, it would also be true if we substitute $(1-G_0)^{-1}$ for EZ in all the three assertions.

Proof: Let $r_1 = r'$ in (1), $r_2 = |S_t|$ in (2), and $r_3 = |N_r \setminus S|$ in (3). We will prove this theorem by induction on r_1, r_2 and r_3; and the absolute summability assumption will justify all the following interchanges of E's, $\int$'s, and $\sum$'s.

Induction basis:

For $r_1 = 1, 2$, (1) is correct. (cf. Iglehart, theorem 5.14.)

For $r_2 = 1$, (2) is correct. (This is a special case of Lemma 4.)

For $r_3 = 1$, (3) is correct. (This is a special case of (2).)

Induction step:

Assuming that (1) is correct up to $r_1 = k < r$, (2) and (3) are correct up to $r_2 = r_3 = k - 1 < r$, then for arbitrary set partition $S_1, \ldots, S_t$ of N_r with $|S_t| = k \leq r$, by using the induction hypothesis, Corollary 1 of Lemma 2, and Lemma 4 (notice that the absolute summability assumption justifies the usage of each of them), we have:

$$L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}$$

$$=\sum_{j=1}^{k-1}\sum_{\substack{S:|S|=j\\ S\subsetneq S_t}} L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S)<(n_i:i\in S_t\setminus S)]}$$

$$+ L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}$$

$$=\sum_{j=1}^{k-1}\sum_{\substack{S:|S|=j\\ S\subsetneq S_t}} A(EZ(\otimes_{i\in S} f_i \circ G_j E\{\prod_{i\in S_t\setminus S} Y(f_i)\})) + A(EZ(\otimes_{i\in S_t} f_i \circ \beta_{|S_t|})),$$

where $A(y)$ is defined as

$$A(y) \equiv EZ(\otimes_{i\in S_1} f_i \circ G_{|S_1|} EZ(\otimes_{i\in S_2} f_i \circ G_{|S_2|} \cdots EZ(\otimes_{i\in S_{t-1}} f_i \circ G_{|S_{t-1}|}\, y)\cdots)).$$

Since A is a linear operator and (1) is correct up to $r_1 = k$, the last equation becomes:

$$L_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}$$

$$=A(E\{\prod_{i\in S_t} Y(f_i)\})$$

$$=EZ(\otimes_{i\in S_1} f_i \circ G_{|S_1|} EZ(\otimes_{i\in S_2} f_i \circ G_{|S_2|} \cdots EZ(\otimes_{i\in S_{t-1}} f_i \circ G_{|S_{t-1}|} E\{\prod_{i\in S_t} Y(f_i)\})\cdots)).$$

This shows that (2) is correct up to $r_2 = k$. Since (3) is a special case of (2), we also have: (3) is correct up to $r_3 = k$.

Now, for $r_1 = k + 1 \le r$,

$$E\prod_{i=1}^{k+1} Y(f_i) = L_{[(n_1,\ldots,n_{k+1})]}$$

$$= \sum_{j=1}^{k} \sum_{\substack{S:|S|=j \\ S\subsetneq N_{k+1}}} L_{[(n_i:i\in S)<(n_i:i\in N_{k+1}\setminus S)]} + L_{[(n_1,\ldots,n_{k+1})]}$$

$$= \sum_{j=1}^{k} \sum_{\substack{S:|S|=j \\ S\subsetneq N_{k+1}}} EZ(\otimes_{i\in S} f_i \circ G_j E\{\prod_{i\in N_{k+1}\setminus S} Y(f_i)\}) + EZ(\otimes_{i=1}^{k+1} f_i \circ \beta_{k+1});$$

Notice that the first two equalities follows from Lemma 2, and the last equality follows from (3) and Lemma 3. We also notice that the usage of each of the above properties is justified by the absolute summability assumption. Since the RHS of the last equality is exactly (1) with $r_1 = k + 1$, we have proved that (1) is correct up to $r_1 = k + 1$. This completes the induction step, and the theorem now follows.

Formula (1) can be simplified when X has a special structure. Note that when X is a CTMC, then $F(x, y, dt) = \lambda(x)\,exp(-\lambda(x)t)\,dt$, for $t > 0$, so that $\mu_n(x, y) = n!/(\lambda(x))^n \equiv \eta_n(x)$. Hence $G_n = G_0 \circ \mu_n = \eta_n \circ G_0$, and $\beta_n = (\mu_n \circ P)e = \eta_n$. We find that (1) can be rewritten as

$$E\{\prod_{i=1}^{r} Y(f_i)\} = \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} EZ(\otimes_{i\in S} f_i \circ \eta_j \circ G_0 E\{\prod_{i\in N_r\setminus S} Y(f_i)\}) + EZ(\otimes_{i=1}^{r} f_i \circ \eta_r)$$

$$= EZ\Big(\sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} \otimes_{i\in S} f_i \circ \eta_j \circ G_0 E\{\prod_{i\in N_r\setminus S} Y(f_i)\} + \otimes_{i=1}^{r} f_i \circ \eta_r\Big).$$

On the other hand, for a DTMC, $\mu_n(x, y) = 1$ for each $x, y \in E$, thus $\beta_n = 1$ and $G_n = G_0$ for each n, so that (1) takes the form:

$$E\{\prod_{i=1}^{r} Y(f_i)\} = \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} EZ(\otimes_{i\in S} f_i \circ G_0 E\{\prod_{i\in N_r\setminus S} Y(f_i)\}) + EZ(\otimes_{i=1}^{r} f_i)$$

$$= EZ\Big(\sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} \otimes_{i\in S} f_i \circ G_0 E\{\prod_{i\in N_r\setminus S} Y(f_i)\} + \otimes_{i=1}^{r} f_i\Big).$$

Notice that when $r \leq 2$, the above two special cases reduce to the formulas given in Hordijk, Iglehart, and Schassberger (1976).

The above formulas are based on the special structure of X. Other reductions are possible, for example, if $f_i = f$, for $i = 1, \ldots, r$, then

$$E(Y(f))^r = \sum_{j=1}^{r-1} \binom{r}{j} EZ(f^j \circ G_j E(Y(f))^{r-j}) + EZ(f^r \circ \beta_r)$$

$$= EZ\left(\sum_{j=1}^{r-1} \binom{r}{j} f^j \circ G_j E(Y(f))^{r-j} + f^r \circ \beta_r\right).$$

On the other hand, if $f_i = g, i = 1, \ldots, m; f_i = h, i = m+1, \ldots, m+n$, then

$$E(Y(g))^m (Y(h))^n$$

$$= \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n \\ 0 < i+j < m+n}} \binom{m}{i} \binom{n}{j} EZ(g^i \circ h^j \circ G_{i+j} E(Y(g))^{m-i} (Y(h))^{n-j}) + EZ(g^m \circ h^n \circ \beta_{m+n})$$

$$= EZ\left(\sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n \\ 0 < i+j < m+n}} \binom{m}{i} \binom{n}{j} g^i \circ h^j \circ G_{i+j} E(Y(g))^{m-i} (Y(h))^{n-j} + g^m \circ h^n \circ \beta_{m+n} \right).$$

Notice that this is exactly the equation (2.3) in Glynn and Iglehart (1984).

4. A recursive formula for discrete time Markov chains

Let $\{X_k : k \geq 0\}$ be an irreducible DTMC with countable state space E, and one step transition matrix $P = \{P_{xy} : x, y \in E\}$. For fixed $z \in E$, let $T(z) = \inf\{n : X_{n-1} \neq z, X_n = z\}$ and set (notice that we will always suppress the z in $T(z)$ when there is no confusion)

$$Y(f) = \sum_{k=0}^{T-1} f(X_k),$$

where $f : E \to R$ is an arbitrary function. We wish to study mixed moments of the form

$$E \prod_{i=1}^{r} Y(f_i),$$

when $f_i : E \to R$ is an arbitrary function for $i = 1, 2 \ldots, r$, and r is a positive integer. We notice that $Y(f) = \sum_{k=0}^{T-1} f(X_k) = \sum_{k=0}^{\infty} f(X_k) 1_{\{T>k\}}$, this is almost of the same form as the decomposition for $Y(f)$ given before, this gives us the motivation to follow the development in the SMP case quite closely. We also need the following modifications of notations:

$$G_0(x, y) = \begin{cases} P_{xy}, & \text{if } y \neq z; \\ 0, & \text{if } y = z. \end{cases}$$

And revise L into L^D (D for DTMC) such as:

$$L_A^D \equiv E\left\{ \sum_{(n_1,\ldots,n_r)\in A} \{\prod_{i=1}^{r} [f_i(X_{n_i}) 1_{\{T>n_i\}}]\} \right\},$$

for each arbitrary set $A \subseteq N_*^r$. Notice that an immediate consequence of this definition is: $E \prod_{i=1}^{r} Y(f_i) = L_{N_*^r}^D$.

Lemma 5.

$$1_{[(n_1,\ldots,n_r)]} = \sum_{j=1}^{r-1} \sum_{\substack{S: |S|=j \\ S \subseteq N_r}} (-1)^{j-1} 1_{[(n_i : i \in S) \leq (n_i : i \in N_r \setminus S)]} + (-1)^{r-1} 1_{[(n_1,\ldots,n_r)]}.$$

Proof: We first notice that

$$[(n_1, \ldots, n_r)] = \bigcup_{i=1}^{r} [n_i \leq (n_j : j \in N_r \setminus \{i\})],$$

since at least one of the element in parameters $\{n_1, \dots, n_r\}$ must be less than or equal to the rest; and each set of RHS is a subset of the set in LHS. Define $W_i \equiv [n_i \leq \langle n_j : j \in N_r \setminus \{i\}\rangle]$, then

$$
\begin{aligned}
&1_{[(n_1,\dots,n_r)]}\\
&=1_{\cup_{i=1}^r W_i}\\
&=\sum_{j=1}^{r} \sum_{\substack{S:|S|=j\\ S\subseteq N_r}} (-1)^{j-1} 1_{\cap_{i\in S} W_i}\\
&=\sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j\\ S\subsetneq N_r}} (-1)^{j-1} 1_{[(n_i:i\in S)\leq(n_i:i\in N_r\setminus S)]} + (-1)^{r-1} 1_{[(n_1,\dots,n_r)]}.
\end{aligned}
$$

This completes the proof.

We now state a sequence of corollarys, lemmas, and theorem, which can be regarded as the "$\leq$" version of their counterparts in Section 2. We will omit all the proofs because of the similarity between their proofs and their counterparts'.

Corollary 1.

$$
\begin{aligned}
&1_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}\\
&=\sum_{j=1}^{|S_t|-1} \sum_{\substack{S:|S|=j\\ S\subsetneq S_t}} (-1)^{j-1} 1_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S)\leq(n_i:i\in S_t\setminus S)]}\\
&\qquad + (-1)^{|S_t|-1} 1_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}.
\end{aligned}
$$

Corollary 2. *If* $|\tilde{f}|_{[(n_1,\dots,n_r)]} = \sum_{(n_1,\dots,n_r)\in N_\bullet^r} |f(n_1,\dots,n_r)| < \infty$ *then*

$$
\tilde{f}_{[(n_1,\dots,n_r)]} = \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j\\ S\subsetneq N_r}} (-1)^{j-1} \tilde{f}_{[(n_i:i\in S)\leq(n_i:i\in N_r\setminus S)]} + (-1)^{r-1} \tilde{f}_{[(n_1,\dots,n_r)]}.
$$

Corollary 3. *If* $|\tilde{f}|_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}<\infty$, *then*

$$\tilde{f}_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}$$

$$=\sum_{j=1}^{|S_t|-1}\sum_{\substack{S:|S|=j\\ S\subsetneq S_t}}(-1)^{j-1}\tilde{f}_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S)\leq(n_i:i\in S_t\setminus S)]}$$

$$+(-1)^{|S_t|-1}\tilde{f}_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}.$$

Lemma 6. *If* $\sum_{(n_1,\ldots,n_r)\in N_r^*}E\{\prod_{i=1}^r|f_i(X_{n_i})|1_{\{T>n_i\}}\}<\infty$ *then*

$$E\prod_{i=1}^r Y(f_i)=L^D_{[(n_1,\ldots,n_r)]}$$

$$=\sum_{j=1}^{r-1}\sum_{\substack{S:|S|=j\\ S\subsetneq N_r}}(-1)^{j-1}L^D_{[(n_i:i\in S)\leq(n_i:i\in N_r\setminus S)]}+(-1)^{r-1}L^D_{[(n_1,\ldots,n_r)]}.$$

Corollary 1. *If*

$$\sum_{(n_1,\ldots,n_r)\in[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S)\leq(n_i:i\in S_t\setminus S)]}E\{\prod_{i=1}^r|f_i(X_{n_i})|1_{\{T>n_i\}}\}<\infty$$

then

$$L^D_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}$$

$$=\sum_{j=1}^{|S_t|-1}\sum_{\substack{S:|S|=j\\ S\subsetneq S_t}}(-1)^{j-1}L^D_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S)\leq(n_i:i\in S_t\setminus S)]}$$

$$+(-1)^{|S_t|-1}L^D_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}.$$

Lemma 7. *If* $\sum_{n=0}^{\infty}E\{|\prod_{i=1}^r|f_i(X_n)||1_{\{T>n\}}\}<\infty$ *then*

$$L^D_{[(n_1,\ldots,n_r)]}=\sum_{n=0}^{\infty}G_0^n(\odot_{i=1}^r f_i)=EZ(\odot_{i=1}^r f_i)=EY(\otimes_{i=1}^r f_i).$$

Moreover, the vector $L^D_{[(n_1,\ldots,n_r)]}$ *is the unique solution,* y, *of*

$$y=\odot_{i=1}^r f_i+G_0 y$$

satisfying $G_0^n y \to 0$ as $n \to \infty$.

Remark: For $r = 1$, this lemma says: $EY(f) = \sum_{n=0}^{\infty} G_0^n f = EZ(f)$. Thus for DTMC, if $G_0^n(\cdot)$ is absolute summable, then $EY(\cdot) = EZ(\cdot)$.

Remark: If E is finite, then $G_0^n \to 0$ as $n \to \infty$, thus the uniqueness is automatically satisfied, and then $L^D_{[(n_1,\ldots,n_r)]} = (1 - G_0)^{-1}[\otimes_{i=1}^r f_i]$.

Lemma 8. *If $\sum_{(n_1,\ldots,n_r)\in[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_t)]} E\{\prod_{i=1}^r |f_i(X_{n_i})| 1_{\{\delta>n_i\}}\} < \infty$ then*

$$L^D_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_t)]}$$

$$= \sum_{n_{S_1}=0}^{\infty} G_0^{n_{S_1}} (\otimes_{i\in S_1} f_i \circ \sum_{n_{S_2}=0}^{\infty} G_0^{n_{S_2}} (\otimes_{i\in S_2} f_i \circ \cdots \circ \sum_{n_{S_t}=0}^{\infty} G_0^{n_{S_t}} (\otimes_{i\in S_t} f_i)\cdots))$$

$$= EZ\Big(\otimes_{i\in S_1} f_i \circ EZ(\otimes_{i\in S_2} f_i \circ \cdots \circ EZ(\otimes_{i\in S_t} f_i)\cdots)\Big)$$

$$= EY\Big(\otimes_{i\in S_1} f_i \circ EY(\otimes_{i\in S_2} f_i \circ \cdots \circ EY(\otimes_{i\in S_t} f_i)\cdots)\Big).$$

Theorem 2. *For a irreducible DTMC, and arbitrary integer r, if $\sum_{(n_1,\ldots,n_r)\in N_r^*} E\{\prod_{i=1}^r |f_i(X_{n_i})| 1_{\{T>n_i\}}\} < \infty$ then,*

(1) for each $r' \leq r$, and $EZ(f) \equiv \sum_{n=0}^{\infty} G_0^n f$, we have

$$E\{\prod_{i=1}^{r'} Y(f_i)\} = L^D_{[(n_1,\ldots,n_{r'})]}$$

$$= \sum_{j=1}^{r'-1} \sum_{\substack{S:|S|=j\\ S\subseteq N_{r'}}} (-1)^{j-1} EZ(\otimes_{i\in S} f_i \circ E\{\prod_{i\in N_{r'}\setminus S} Y(f_i)\}) + (-1)^{r'-1} EZ(\otimes_{i=1}^{r'} f_i)$$

$$= EZ\Big(\sum_{j=1}^{r'-1} \sum_{\substack{S:|S|=j\\ S\subseteq N_{r'}}} (-1)^{j-1} \otimes_{i\in S} f_i \circ E\{\prod_{i\in N_{r'}\setminus S} Y(f_i)\} + (-1)^{r'-1} \otimes_{i=1}^{r'} f_i\Big);$$

(2) for arbitrary partition $S_1, \ldots, S_t (t \geq 2)$ of N_r,

$$L^D_{[(n_i:i\in S_1)\leq\cdots\leq(n_i:i\in S_{t-1})\leq(n_i:i\in S_t)]}$$

$$= EZ(\otimes_{i\in S_1} f_i \circ EZ(\otimes_{i\in S_2} f_i \circ \cdots \circ EZ(\otimes_{i\in S_{t-1}} f_i \circ E\{\prod_{i\in S_t} Y(f_i)\})\cdots));$$

(3) for arbitrary nonempty proper subset S of N_r,

$$L^D_{[(n_i:i\in S)\leq(n_i:i\in N\setminus S)]} = EZ(\otimes_{i\in S} f_i \circ E\{\prod_{i\in N\setminus S} Y(f_i)\}).$$

Remark: Since $EZ(\cdot) = EY(\cdot)$ if any of them converges absolutely, under the condition of this theorem, each assertion is correct if we substitute EY into EZ.

Formula (1) can be simplified when f_i's have some special structure, for example, if $f_i = f$, for $i = 1,\ldots,r$, then

$$\begin{aligned}E(Y(f))^r &= \sum_{j=1}^{r-1}\binom{r}{j}(-1)^{j-1}EZ(f^j\circ E(Y(f))^{r-j}) + (-1)^{r-1}EZ(f^r)\\ &= EZ\left(\sum_{j=1}^{r-1}\binom{r}{j}(-1)^{j-1}f^j\circ E(Y(f))^{r-j} + (-1)^{r-1}f^r\right)\\ &= EY\left(\sum_{j=1}^{r-1}\binom{r}{j}(-1)^{j-1}f^j\circ E(Y(f))^{r-j} + (-1)^{r-1}f^r\right).\end{aligned}$$

Notice that this is exactly the equation (2.6) in Glynn and Iglehart (1984). On the other hand, if $f_i = g, i = 1,\ldots,m$; $f_i = h, i = m+1,\ldots,m+n$, then

$$\begin{aligned}&E(Y(g))^m(Y(h))^n\\ &= \sum_{\substack{0\leq i\leq m\\ 0\leq j\leq n\\ 0<i+j<m+n}}\binom{m}{i}\binom{n}{j}(-1)^{i+j-1}EZ(g^i\circ h^j\circ E(Y(g))^{m-i}(Y(h))^{n-j}) + (-1)^{m+n-1}EZ(g^m\circ h^n)\\ &= EZ\left(\sum_{\substack{0\leq i\leq m\\ 0\leq j\leq n\\ 0<i+j<m+n}}\binom{m}{i}\binom{n}{j}(-1)^{i+j-1}g^i\circ h^j\circ E(Y(g))^{m-i}(Y(h))^{n-j} + (-1)^{m+n-1}g^m\circ h^n\right)\\ &= EY\left(\sum_{\substack{0\leq i\leq m\\ 0\leq j\leq n\\ 0<i+j<m+n}}\binom{m}{i}\binom{n}{j}(-1)^{i+j-1}g^i\circ h^j\circ E(Y(g))^{m-i}(Y(h))^{n-j} + (-1)^{m+n-1}g^m\circ h^n\right).\end{aligned}$$

5. A recursive formula for continuous time Markov chains

We will show how to get another recursive moment formula for CTMC in this section. First, let $X = \{X(t) : t \geq 0\}$ be a CTMC with countable state space E, transition matrix $P(t) = \{P_{xy}(t) : x, y \in E\}$, and Q-matrix $Q = \{q_{xy} : x, y \in E\}$ as the infinitesimal transition parameters. Recall that in a continuous time case $Q = P'(0)$ is the given data of the model and that $P(t)$ is generally hard to calculate and rarely given explicitly. The exponential distribution holding time in any state $x \in E$ has mean $q_x^{-1} \equiv q_{xx}^{-1}$. For all $x \in E$, we assume that $0 < q_x < \infty$, so that all states are stable and nonabsorbing. In addition, we assume that $\sum_{y \in E} q_{xy} = 0$, which guarantees that, starting from any state $x \in E$, the CTMC makes a transition to a next state $y \in E$. The element of the jump matrix $G = \{G(x, y) : x, y \in E\}$ of X are defined by

$$G(x,y) = \begin{cases} q_{xy}/q_x, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

We will assume that G is irreducible. Notice that this is equivalent to X is irreducible, and therefore, positive recurrent. For fixed $z \in E$ as the regenerative state, let $T(z) \equiv \inf\{t > 0 : X(t-) \neq z, X(t) = z\}$ and

$$Y(f) = \int_0^{T(z)} f(X(t))\, dt,$$

where $f : E \to R$ is an arbitrary function. (Hereafter, we will suppress the z in $T(z)$ when no confusion is likely.) Our objective is to study the mixed moments of the form

$$E \prod_{i=1}^{r} Y(f_i),$$

when $f_i : E \to R$ is an arbitrary function, for $i = 1, 2, \ldots, r$, and r is a positive integer.

We need to define, as in Iglehart [3], ${}_0P_{xy}(t) = P_x\{T > t, X(t) = y\}$, and ${}_0P(t) = \{{}_0P_{xy}(t) : x, y \in E\}$; also let

$$G_0(x,y) = \begin{cases} G(x,y), & \text{if } y \neq z; \\ 0, & \text{if } y = z. \end{cases}$$

And let $\sigma \equiv (\sigma_1, \sigma_2, \ldots, \sigma_r)$ to be a permutation of $(1, 2, \ldots, r)$; $\sigma^j \equiv (j, \sigma_{j2}, \sigma_{j3}, \ldots, \sigma_{jr})$, where $(\sigma_{j2}, \ldots, \sigma_{jr})$ is a permutation of $(1, \ldots, j-1, j+1, \ldots, r)$; and $\bar{\sigma}^j \equiv (\bar{\sigma}_{j1}, \ldots, \bar{\sigma}_{j,r-1})$ to be a permutation of $(1, \ldots, j-1, j+1, \ldots, r)$.

From the above definition, we immediately have

$$\{\sigma\} = \bigcup_{j=1}^{r} \{\sigma^j\},$$

which means: the set of all permutations of $(1,\ldots,r)$ is the union (over all j's) of the sets of all permutations beginning in j. As a consequence

$$\begin{aligned}R_+^r &\equiv \{(w_1,\ldots,w_r): 0 \le w_i < \infty, i = 1,\ldots,r\}\\ &= \bigcup_\sigma \{(w_1,\ldots,w_r) \in R_+^r : w_{\sigma_1} \le w_{\sigma_2} \le \cdots \le w_{\sigma_r}\}\\ &= \bigcup_{j=1}^r \bigcup_{\sigma_j} \{(w_1,\ldots,w_r) \in R_+^r : w_j \le w_{\sigma_{j_2}} \le \cdots \le w_{\sigma_{j_r}}\}.\end{aligned}$$

We also notice that the sets in the RHS are almost mutually exclusive, in the sence that each pair of sets have an intersection which is Lebesgue measure 0.

We begin by citing a theorem from Hordijk, Iglehart, and Schassberger (1976).

Lemma 9. *(Hordijk, Iglehart, and Schassberger (1976).)*

$$EY(f) = \int_0^\infty {}_0P(t) f\,dt = \sum_{n=0}^\infty G_0^n(f \circ q^{-1}) = EZ(f \circ q^{-1}).$$

provided that the integral (or equivalently, the sum) converges absolutely.

Lemma 10. *If* $\int_{w_{\sigma_1} \le \quad \le w_{\sigma_r}} E\{\prod_{i=1}^r |f_{\sigma_i}(X(w_{\sigma_i}))| 1_{\{T > w_{\sigma_i}\}}\} \, dw_{\sigma_1} \cdots dw_{\sigma_r} < \infty$, *then*

$$\begin{aligned}&E \int_{w_{\sigma_1} \le \quad \le w_{\sigma_r}} \{\prod_{i=1}^r f_{\sigma_i}(X(w_{\sigma_i})) 1_{\{T > w_{\sigma_i}\}}\} \, dw_{\sigma_1} \cdots dw_{\sigma_r}\\ &= \sum_{n_1=0}^\infty G_0^{n_1} f_{\sigma_1} \circ q^{-1} \circ \sum_{n_2=0}^\infty G_0^{n_2} f_{\sigma_2} \circ q^{-1} \circ \cdots \circ \sum_{n_r=0}^\infty G_0^{n_r} f_{\sigma_r} \circ q^{-1}\\ &= EZ(f_{\sigma_1} \circ q^{-1} \circ EZ(f_{\sigma_2} \circ q^{-1} \circ \cdots \circ EZ(f_{\sigma_r} \circ q^{-1}) \cdots))\\ &= EY(f_{\sigma_1} \circ EY(f_{\sigma_2} \circ \cdots \circ EY(f_{\sigma_r}) \cdots)),\end{aligned}$$

Proof:

$$E_l \int_{w_{\sigma_1} \leq \cdots \leq w_{\sigma_r}} \{\prod_{i=1}^{r} f_{\sigma_i}(X(w_{\sigma_i})) 1_{\{T > w_{\sigma_i}\}}\} dw_{\sigma_1} \cdots dw_{\sigma_r}$$

$$= \int_{w_{\sigma_1} \leq \cdots \leq w_{\sigma_r}} E_l\{\prod_{i=1}^{r} f_{\sigma_i}(X(w_{\sigma_i})) 1_{\{T > w_{\sigma_i}\}}\} dw_{\sigma_1} \cdots dw_{\sigma_r}$$

$$= \int_{w_{\sigma_1}=0}^{\infty} \int_{w_{\sigma_2} \geq w_{\sigma_1}} \cdots \int_{w_{\sigma_r} \geq w_{\sigma_{r-1}}} E_l\{\prod_{i=1}^{r} f_{\sigma_i}(X(w_{\sigma_i})) 1_{\{T > w_{\sigma_i}\}}\} dw_{\sigma_1} \cdots dw_{\sigma_r}$$

$$= \int_{w_{\sigma_1}=0}^{\infty} \int_{w_{\sigma_2}=0}^{\infty} \cdots \int_{w_{\sigma_r}=0}^{\infty} E_l\{\prod_{i=1}^{r} f_{\sigma_i}(X(\sum_{j=1}^{i} w_{\sigma_j})) 1_{\{T > \sum_{j=1}^{i} w_{\sigma_j}\}}\} dw_{\sigma_1} \cdots dw_{\sigma_r}$$

$$= \int_{w_{\sigma_1}=0}^{\infty} \int_{w_{\sigma_2}=0}^{\infty} \cdots \int_{w_{\sigma_r}=0}^{\infty} E_l\{ \sum_{j_1 \in E} f_{\sigma_1}(j_1) 1_{\{T > w_{\sigma_1}\}} 1_{\{X(w_{\sigma_1}) = j_1\}}$$

$$\sum_{j_2 \in E} f_{\sigma_2}(j_2) 1_{\{T > w_{\sigma_1} + w_{\sigma_2}\}} 1_{\{X(w_{\sigma_1} + w_{\sigma_2}) = j_2\}} \cdots$$

$$\sum_{j_r \in E} f_{\sigma_r}(j_r) 1_{\{T > w_{\sigma_1} + \cdots + w_{\sigma_r}\}} 1_{\{X(w_{\sigma_1} + \cdots + w_{\sigma_r}) = j_r\}}\} dw_{\sigma_1} \cdots dw_{\sigma_r}$$

$$= \int_{w_{\sigma_1}=0}^{\infty} \int_{w_{\sigma_2}=0}^{\infty} \cdots \int_{w_{\sigma_r}=0}^{\infty} \sum_{j_1 \in E} f_{\sigma_1}(j_1) {}_0P_{l j_1}(w_{\sigma_1}) \sum_{j_2 \in E} f_{\sigma_2}(j_2) {}_0P_{j_1 j_2}(w_{\sigma_2})$$

$$\cdots \sum_{j_r \in E} f_{\sigma_r}(j_r) {}_0P_{j_{r-1} j_r}(w_{\sigma_r}) dw_{\sigma_1} \cdots dw_{\sigma_r}$$

$$= \int_{w_{\sigma_1}=0}^{\infty} \sum_{j_1 \in E} f_{\sigma_1}(j_1) {}_0P_{l j_1}(w_{\sigma_1}) dw_{\sigma_1} \int_{w_{\sigma_2}=0}^{\infty} \sum_{j_2 \in E} f_{\sigma_2}(j_2) {}_0P_{j_1 j_2}(w_{\sigma_2}) dw_{\sigma_2}$$

$$\cdots \int_{w_{\sigma_r}=0}^{\infty} \sum_{j_r \in E} f_{\sigma_r}(j_r) {}_0P_{j_{r-1} j_r}(w_{\sigma_r}) dw_{\sigma_r}$$

$$= \sum_{n_1=0}^{\infty} \left[G_0^{n_1} f_{\sigma_1} \circ q^{-1} \circ \sum_{n_2=0}^{\infty} G_0^{n_2} f_{\sigma_2} \circ q^{-1} \circ \cdots \circ \sum_{n_r=0}^{\infty} G_0^{n_r} f_{\sigma_r} \circ q^{-1}\right]_l ,$$

where our absolute integrability assumption justifies the various interchanges of E's, $\int$'s, and $\sum$'s; and Lemma 9 is used r times to obtain the last equality.

This completes the proof.

Theorem 3. *If* $\int_{R_+^r} E\{\prod_{i=1}^{r} |f_{\sigma_i}(X(w_{\sigma_i}))| 1_{\{T > w_{\sigma_i}\}}\} dw_{\sigma_1} \cdots dw_{\sigma_r} < \infty$, *then*

$$E \prod_{i=1}^{r'} Y(f_i) = E \sum_{\sigma} \int_{w_{\sigma_1} \leq \cdots \leq w_{\sigma_{r'}}} \{\prod_{i=1}^{r'} f_{\sigma_i}(X(w_{\sigma_i})) 1_{\{T > w_{\sigma_i}\}}\} dw_{\sigma_1} \cdots dw_{\sigma_{r'}}$$

$$= \sum_{i=1}^{r'} \Big[\sum_{n=0}^{\infty} G_0^n f_i \circ q^{-1} \circ E \prod_{j \neq i} Y(f_j) \Big]$$

$$= \sum_{n=0}^{\infty} G_0^n \Big[\sum_{i=1}^{r'} f_i \circ q^{-1} \circ E \prod_{j \neq i} Y(f_j) \Big]$$

$$= EZ \Big[\sum_{i=1}^{r'} f_i \circ q^{-1} \circ E \prod_{j \neq i} Y(f_j) \Big]$$

$$= EY \Big[\sum_{i=1}^{r'} f_i \circ E \prod_{j \neq i} Y(f_j) \Big],$$

for any $r' \leq r$.

Remark: When $r = 2$, the theorem reduces to

$$EY(f_1)Y(f_2) = \sum_{n=0}^{\infty} G_0^n \big[f_1 \circ q^{-1} \circ EY(f_2) + f_2 \circ q^{-1} \circ EY(f_1) \big].$$

This agrees with equation (3.13) in Hordijk, Iglehart, and Schassberger (1976).

Proof: We prove this theorem by induction.

Induction basis:

From Hordijk, Iglehart, and Schassberger (1976), the assertion is correct for $r' = 1$.

Induction step:

Suppose the assertion is correct up to $r' = k - 1 < r$, consider

$$E \prod_{i=1}^{k} Y(f_i)$$

$$= E \int_{R_+} f_1(X(w_1)) 1_{\{T > w_1\}} \, dw_1 \int_{R_+} f_2(X(w_2)) 1_{\{T > w_2\}} \, dw_2$$

$$\cdots \int_{R_+} f_k(X(w_k)) 1_{\{T > w_k\}} \, dw_k$$

$$= E \int_{R_+^k} \{ \prod_{i=1}^{k} f_i(X(w_i)) 1_{\{T > w_i\}} \} \, dw_1 \cdots dw_k$$

$$= E \sum_{j=1}^{k} \sum_{\sigma_j} \int_{j \le w_{\sigma_{j,2}} \le \cdots \le w_{\sigma_{j,k}}} \{ \prod_{i=1}^{k} f_i(X(w_i)) 1_{\{T > w_i\}} \} \, dw_1 \cdots dw_k$$

$$= \sum_{j=1}^{k} \sum_{\sigma_j} E \int_{j \le w_{\sigma_{j,2}} \le \cdots \le w_{\sigma_{j,k}}} \{ \prod_{i=1}^{k} f_i(X(w_i)) 1_{\{T > w_i\}} \} \, dw_1 \cdots dw_k$$

$$= \sum_{j=1}^{k} \sum_{\sigma_j} \Big[\sum_{n_1=0}^{\infty} G_0^{n_1} f_j \circ q^{-1} \circ \sum_{n_2=0}^{\infty} G_0^{n_2} f_{\sigma_{j,2}} \circ q^{-1} \circ \cdots \circ \sum_{n_r=0}^{\infty} G_0^{n_k} f_{\sigma_{j,k}} \circ q^{-1} \Big]$$

$$= \sum_{j=1}^{k} \sum_{n_1=0}^{\infty} G_0^{n_1} f_j \circ q^{-1} \circ \Big[\sum_{\overline{\sigma}_j} \sum_{n_2=0}^{\infty} G_0^{n_2} f_{\sigma_{j,2}} \circ q^{-1} \circ \cdots \circ \sum_{n_r=0}^{\infty} G_0^{n_k} f_{\sigma_{j,k}} \circ q^{-1} \Big]$$

$$= \sum_{j=1}^{k} \Big[\sum_{n=0}^{\infty} G_0^{n} f_j \circ q^{-1} \circ E \prod_{i \ne j} Y(f_i) \Big]$$

$$= \sum_{n=0}^{\infty} G_0^{n} \Big[\sum_{j=1}^{k} f_j \circ q^{-1} \circ E \prod_{i \ne j} Y(f_i) \Big] \quad ,$$

where the absolute integrability assumption justifies the various interchanges of $\sum$'s, $\int$'s, and E's; Lemma 10 justifies the fifth equality; and the last equation is obtained by the induction hypothesis.

This completes the induction step, and the theorem follows.

6. Generalisation and discussion

We will discuss some generalizations in this section. As before, let $X = \{X(t) : t \geq 0\}$ be an irreducible semi-Markov process (SMP) on countable state space E, and $R = \{R_n : n \geq 0\}$ be the underlying Markov chain of X. For each t, denote $X^*(t) \equiv X(t + t^*)$, where $t^* = \inf\{t' \geq 0 : X(t + t') \neq X(t)\}$; namely, $X^*(t)$ is defined as the "next state" process of X at time t. For fixed $z \in E$, let $T(z) = \inf\{t \geq 0 : X(t-) \neq z, X(t) = z\}$, $\delta(z) = \inf\{n \geq 0 : R_{n-1} \neq z, R_n = z\}$ and set

$$Y'(f') = \int_0^{T(z)} f'(X(t), X^*(t))\, dt,$$

where $f' : E \times E \to R$ is an arbitrary function. (Hereafter, we will suppress the z in $T(z)$ and $\delta(z)$ when no confusion is likely.) Our first objective is to study the mixed moments of the form $E \prod_{i=1}^{r} Y'(f_i')$, when $f_i' : E \times E \to R$ are arbitrary functions, for $i = 1, 2, \ldots, r$, and r is a positive integer.

We begin by noting

$$\begin{aligned}
Y'(f') &= \int_0^{T(z)} f'(X(t), X^*(t))\, dt \\
&= \int_0^{\infty} f'(X(t), X^*(t))\, 1_{\{T>t\}}\, dt \\
&= \sum_{n=0}^{\infty} \int_{\xi_n}^{\xi_{n+1}} f'(X(t), X^*(t))\, 1_{\{T>t\}}\, dt \\
&= \sum_{n=0}^{\infty} f'(R_n, R_{n+1})\, 1_{\{\delta>n\}}\, (\xi_{n+1} - \xi_n),
\end{aligned}$$

it follows that $\prod_{i=1}^{r} Y'(f_i') = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \prod_{i=1}^{r} f_i'(R_{n_i}, R_{n_i+1})(\xi_{n_i+1} - \xi_{n_i}) 1_{\{\delta>n_i\}}$. Next, we define L' as follows: for fixed f_i, $i = 1, \ldots, r$, and fixed $A \subseteq N_+^r$, let L' be

$$L'_A \equiv E\left\{ \sum_{(n_1,\ldots,n_r)\in A} \left[\prod_{i=1}^{r} f_i'(R_{n_i}, R_{n_i+1})(\xi_{n_i+1} - \xi_{n_i}) 1_{\{\delta>n_i\}}\right]\right\},$$

where δ is the length of the first z-cycle for the embedded DTMC R.

We proceed via a series of lemmas. Note that the proofs for them are very similar to the proofs for their counterparts in Section 3 and thus omitted here.

Lemma 11. *If* $\sum_{(n_1,\dots,n_r)\in N_r} E\{\prod_{i=1}^r |f_i'(R_{n_i}, R_{n_i+1})|(\xi_{n_i+1} - \xi_{n_i})1_{\{\delta > n_i\}}\} < \infty$ *then*

$$E\prod_{i=1}^r Y'(f_i') = L'_{[(n_1,\dots,n_r)]}$$

$$= \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subseteq N_r}} L'_{[(n_i:i\in S)<(n_i:i\in N_r\setminus S)]} + L'_{[(n_1,\dots,n_r)]}.$$

Corollary 1. *If*

$$\sum_{(n_1,\dots,n_r)\in[(n_i:i\in S_1)< \cdots <(n_i:i\in S_{t-1})<(n_i:i\in S_t)]} E\{\prod_{i=1}^r |f_i'(R_{n_i}, R_{n_i+1})|(\xi_{n_i+1} - \xi_{n_i})1_{\{\delta > n_i\}}\} < \infty$$

then

$$L'_{[(n_i:i\in S_1)< \cdots <(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}$$

$$= \sum_{j=1}^{|S_t|-1} \sum_{\substack{S:|S|=j \\ S\subseteq S_t}} L'_{[(n_i:i\in S_1)< \cdots <(n_i:i\in S_{t-1})<(n_i:i\in S)<(n_i:i\in S_t\setminus S)]}$$

$$+ L'_{[(n_i:i\in S_1)< \cdots <(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}.$$

Lemma 12. *If* $\sum_{n=0}^\infty E\{|\prod_{i=1}^r |f_i'(R_n, R_{n+1})||(\xi_{n+1} - \xi_n)^r 1_{\{\delta > n\}}\} < \infty$, *and define* $EZ(g) \equiv \sum_{n=0}^\infty G_0^n g$, *then*

$$L'_{[(n_1,\dots,n_r)]} = \sum_{n=0}^\infty G_0^n\left[(\otimes_{i=1}^r f_i' \circ \mu_r \circ P)e\right] = EZ\left[(\otimes_{i=1}^r f_i' \circ \mu_r \circ P)e\right].$$

Moreover, the vector $L'_{[(n_1,\dots,n_r)]}$ *is the unique solution,* y, *of*

$$y = (\otimes_{i=1}^r f_i' \circ \mu_r \circ P)e + G_0 y$$

satisfying $G_0^n y \to 0$ *as* $n \to \infty$.

Remark: A consequence of this lemma: If $\sum_{n=0}^\infty E\{|f'(R_n, R_{n+1})|(\xi_{n+1} - \xi_n)1_{\{\delta>n\}}\} < \infty$, then $EY'(f') = L'_{[n]} = \sum_{n=0}^\infty G_0^n[(f' \circ \mu_1 \circ P)e] = EZ\left[(f' \circ \mu_1 \circ P)e\right]$.

Remark: We notice that by setting $f'(x,y) = f(x)$ for all $y \in E$, then $(f' \circ \mu_1 \circ P)e = f \circ \beta_1$. Thus we immediately have another consequence of this lemma: If $\sum_{n=0}^\infty E\{|f(R_n)|(\xi_{n+1} - \xi_n)1_{\{\delta>n\}}\} < \infty$, then $EY(f) = \sum_{n=0}^\infty G_0^n(f \circ \beta_1) = EZ(f \circ \beta_1)$. This is the first part of Theorem (5.14) in Iglehart [3].

Remark: If E is finite, then $G_0^n \to 0$ as $n \to \infty$, thus the uniqueness is automatically satisfied, and also $EZ = \sum_{n=0}^{\infty} G_0^n = (1-G_0)^{-1}$, thus $L'_{[(n_1,\ldots,n_r)]} = (1-G_0)^{-1}\big[(\otimes_{i=1}^{r} f_i' \circ \mu_r \circ P)e\big]$.

Lemma 13. *If* $\sum_{(n_1,\ldots,n_r)\in[(n_i:i\in S_1)<\cdots<(n_i:i\in S_t)]} E\{\prod_{i=1}^{r}[|f_i'(R_{n_i},R_{n_i+1})|(\xi_{n_i+1}-\xi_{n_i})1_{\{\delta>n_i\}}]\} < \infty$ *then,*

$$L'_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_t)]}$$

$$= \sum_{n_{S_1}=0}^{\infty} G_0^{n_{S_1}}\Big[(\otimes_{i\in S_1} f_i' \circ G_{|S_1|}) \sum_{n_{S_2}=0}^{\infty} G_0^{n_{S_2}}\big[(\otimes_{i\in S_2} f_i' \circ G_{|S_2|}) \cdots \sum_{n_{S_t}=0}^{\infty} G_0^{n_{S_t}}[(\otimes_{i\in S_t} f_i' \circ \mu_{|S_t|} \circ P)e] \cdots\big]\Big]$$

$$= EZ\Big[(\otimes_{i\in S_1} f_i' \circ G_{|S_1|}) EZ\big[(\otimes_{i\in S_2} f_i' \circ G_{|S_2|}) \cdots EZ[(\otimes_{i\in S_t} f_i' \circ \mu_{|S_t|} \circ P)e] \cdots\big]\Big].$$

Remark: Lemma 12 is a special case of Lemma 13.

Remark: If E is finite, then $L'_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_t)]} = (1-G_0)^{-1}\big[(\otimes_{i\in S_1} f_i' \circ G_{|S_1|})(1-G_0)^{-1}[(\otimes_{i\in S_2} f_i' \circ G_{|S_2|}) \cdots (1-G_0)^{-1}[(\otimes_{i\in S_t} f_i' \circ \mu_{|S_t|} \circ P)e] \cdots]\big]$.

Theorem 4. *For an irreducible SMP, and arbitrary integer r, if*

$$\sum_{(n_1,\ldots,n_r)\in N_r^<} E\{\prod_{i=1}^{r}[|f_i'(R_{n_i},R_{n_i+1})|(\xi_{n_i+1}-\xi_{n_i})1_{\{\delta>n_i\}}]\} < \infty$$

then,

(1) for each $r' \le r$, and $EZ(g) \equiv \sum_{n=0}^{\infty} G_0^n g$, we have

$$E\{\prod_{i=1}^{r'} Y'(f_i')\} = L'_{[(n_1,\ldots,n_{r'})]}$$

$$= \sum_{j=1}^{r'-1} \sum_{\substack{S:|S|=j\\ S\subseteq N_{r'}}} EZ\Big[(\otimes_{i\in S} f_i' \circ G_j) E\{\prod_{i\in N_{r'}\setminus S} Y'(f_i')\}\Big] + EZ\Big[(\otimes_{i=1}^{r'} f_i' \circ \mu_{r'} \circ P)e\Big]$$

$$= EZ\Big(\sum_{j=1}^{r'-1} \sum_{\substack{S:|S|=j\\ S\subseteq N_{r'}}} (\otimes_{i\in S} f_i' \circ G_j) E\{\prod_{i\in N_{r'}\setminus S} Y'(f_i')\} + (\otimes_{i=1}^{r'} f_i' \circ \mu_{r'} \circ P)e\Big);$$

(2) for arbitrary partition $S_1,\ldots,S_t (t \ge 2)$ of N_r,

$$L'_{[(n_i:i\in S_1)<\cdots<(n_i:i\in S_{t-1})<(n_i:i\in S_t)]}$$

$$= EZ\Big[(\otimes_{i\in S_1} f_i' \circ G_{|S_1|}) EZ\big[(\otimes_{i\in S_2} f_i' \circ G_{|S_2|}) \cdots EZ[(\otimes_{i\in S_{t-1}} f_i' \circ G_{|S_{t-1}|}) E\{\prod_{i\in S_t} Y'(f_i')\}] \cdots\big]\Big];$$

(3) for arbitrary nonempty proper subset S of $\mathcal{N}_r$,

$$L'_{[(n_i:i\in S)<(n_i:i\in \mathcal{N}_r\setminus S)]} = EZ\Big[(\otimes_{i\in S} f'_i \circ G_{|S|})E\{\prod_{i\in N_r\setminus S} Y'(f'_i)\}\Big].$$

Remark: If $r'=1$, (1) becomes: $EY'(f') = EZ[(f'\circ\mu_1\circ P)e] = \sum_{n=0}^{\infty} G_0^n[(f'\circ\mu_1\circ P)e]$. This agrees with Lemma 12.

Remark: If E is finite, in addition to the existing assertions, it would also be true if we substitute $(1-G_0)^{-1}$ for EZ in all the three assertions.

Corollary 1. *For an irreducible SMP, and arbitrary integer r, if*

$$\sum_{(n_1,\ldots,n_r)\in \mathcal{N}^r_{\ne}} E\{\prod_{i=1}^{r}[|f_i(R_{n_i})|(\xi_{n_i+1}-\xi_{n_i})1_{\{\delta>n_i\}}]\}<\infty$$

then for each $r'\le r$, we have

$$E\{\prod_{i=1}^{r'} Y(f_i)\} = \sum_{j=1}^{r'-1}\sum_{\substack{S:|S|=j\\ S\subsetneq N_{r'}}} EZ\Big[(\otimes_{i\in S} f_i\circ G_j)E\{\prod_{i\in N_{r'}\setminus S} Y(f_i)\}\Big] + EZ\Big[\otimes_{i=1}^{r'} f_i\circ\beta_{r'}\Big]$$

$$= EZ\Big(\sum_{j=1}^{r'-1}\sum_{\substack{S:|S|=j\\ S\subsetneq N_{r'}}}(\otimes_{i\in S} f_i\circ G_j)E\{\prod_{i\in N_{r'}\setminus S} Y(f_i)\} + \otimes_{i=1}^{r'} f_i\circ\beta_{r'}\Big);$$

Remark: This is exatly Theorem 1.

Corollary 2. *For an irreducible SMP, and arbitrary integer r, if*

$$\sum_{(n_1,\ldots,n_r)\in \mathcal{N}^r_{\ne}} E\{\prod_{i=1}^{r}[|c_i(R_{n_i},R_{n_i+1})|1_{\{\delta>n_i\}}]\}<\infty,$$

where $Y''(c)\equiv\sum_{n=0}^{\delta-1} c(R_n,R_{n+1})$, and $c, c_i : E\times E\to R$, are arbitrary funtions for $i=1,\ldots,r$, then for each $r'\le r$, we have

$$E\{\prod_{i=1}^{r'} Y''(c_i)\} = \sum_{j=1}^{r'-1}\sum_{\substack{S:|S|=j\\ S\subsetneq N_{r'}}} EZ\Big[(\otimes_{i\in S} c_i\circ G_0)E\{\prod_{i\in N_{r'}\setminus S} Y''(c_i)\}\Big] + EZ\Big[(\otimes_{i=1}^{r'} c_i\circ P)e\Big]$$

$$= EZ\Big(\sum_{j=1}^{r'-1}\sum_{\substack{S:|S|=j\\ S\subsetneq N_{r'}}}(\otimes_{i\in S} c_i\circ G_0)E\{\prod_{i\in N_{r'}\setminus S} Y''(c_i)\} + (\otimes_{i=1}^{r'} c_i\circ P)e\Big);$$

A further generalization of Theorem 4 is possible. Let $F_i : E \times E \to R$ be a random matrix for each $i = 1, \ldots, r$, and let $E[F_i] \equiv \overline{F_i}$, $E[F_i \circ F_j] \equiv \overline{F_i \circ F_j}$, $E[\otimes_i F_i] \equiv \overline{\otimes_i F_i}$, etc. By following the same proedure as in Theorem 4, we will have:

Corollary 3. *For an irreducible SMP, and arbitrary integer r, if F_i's are independent of the holding time $\xi_{n+1} - \xi_n$ for each n, and*

$$\sum_{(n_1,\ldots,n_r)\in N_r^*} E\{\prod_{i=1}^{r} [|\overline{F_i(R_{n_i}, R_{n_i+1})}|(\xi_{n_i+1} - \xi_{n_i})1_{\{\delta > n_i\}}]\} < \infty$$

then,

$$E\{\prod_{i=1}^{r} Y'(F_i)\} = L'_{[(n_1,\ldots,n_r)]}$$

$$= \sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} EZ\left[(\overline{\otimes_{i\in S} F_i} \circ G_j)E\{\prod_{i\in N_r\setminus S} Y'(F_i)\}\right] + EZ\left[(\overline{\otimes_{i=1}^{r} F_i} \circ \mu_r \circ P)e\right]$$

$$= EZ\left(\sum_{j=1}^{r-1} \sum_{\substack{S:|S|=j \\ S\subsetneq N_r}} (\overline{\otimes_{i\in S} F_i} \circ G_j)E\{\prod_{i\in N_r\setminus S} Y'(F_i)\} + (\overline{\otimes_{i=1}^{r} F_i} \circ \mu_r \circ P)e\right);$$

Finally, denote $T_1 \wedge T_2 \equiv min(T_1, T_2)$, then for arbitrary $z_1, z_2 \in E$, $E\int_0^{T(z_1)\wedge T(z_2)} f'(X(t), X^*(t))\,dt$ can be computed in exatly the same way as in Theorem 4 by redefining G_n, for each n, as:

$$G_n(x,y) = \begin{cases} P_{xy}\mu_n(x,y), & \text{if } y \neq z_1 \text{ or } z_2; \\ 0, & \text{if } y = z_1 \text{ or } z_2. \end{cases}$$

Further generalization along this idea is obvious.

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